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Triangulating the Real Projective Plane

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Abstract: We consider the problem of computing a triangulation of the real projective plane $\mathbb{P}^2$, given a finite point set $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$ as input. We prove that a triangulation of $\mathbb{P}^2$ always exists if at least six points in $\mathcal{P}$ are in *general position*, i.e., no three of them are collinear. We also design an algorithm for triangulating $\mathbb{P}^2$ if this necessary condition holds. As far as we know, this is the first computational result on the real projective plane.

Key-words: Computational geometry, triangulation, simplicial complex, projective geometry, algorithm

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Trianguler le plan projectif réel

Résumé : Nous considérons le calcul de la triangulation d'un ensemble fini de points $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$ dans le plan projectif. Nous démontrons que la triangulation existe toujours dès lors qu'au moins six points de $\mathcal{P}$ sont en position générale, c'est-à-dire que trois d'entre eux ne sont jamais alignés. Nous proposons également un algorithme pour trianguler $\mathbb{P}^2$ si cette condition nécessaire est remplie. À notre connaissance, c'est le premier résultat algorithmique connu pour le plan projectif réel.

Mots-clés : Géométrie algorithmique, triangulation, complexe simplicial, géométrie projective, algorithme

1 Introduction

The real projective plane $\mathbb{P}^2$ is in one-to-one correspondence with the set of lines of the vector space $\mathbb{R}^3$. Formally, $\mathbb{P}^2$ is the quotient $\mathbb{P}^2 = \mathbb{R}^3 - \{0\} \ / \sim$ where the equivalence relation $\sim$ is defined as follows: for two points p and p' of $\mathbb{P}^2$, $p \sim p'$ if $p = \lambda p'$ for some $\lambda \in \mathbb{R} - \{0\}$.

Triangulations of the real projective plane $\mathbb{P}^2$ have been studied quite well in the past, though mainly from a graph-theoretic perspective. A *contraction* of and edge e in a map $\mathcal{M}$ removes e and identifies its two endpoints, if the graph obtained by this operation is simple. $\mathcal{M}$ is *irreducible* if none of its edges can be contracted. Barnette [1] proved that the real projective plane admits exactly two irreducible triangulations, which are the complete graph K_6 with six vertices and $K_4 + \overline{K}_3$ (i.e., the quadrangulation by K_4 with each face subdivided by a single vertex), which are shown in Figure 1. Note that these figures are just graphs, i.e. the horizontal and vertical lines do not imply collinearity of the points.

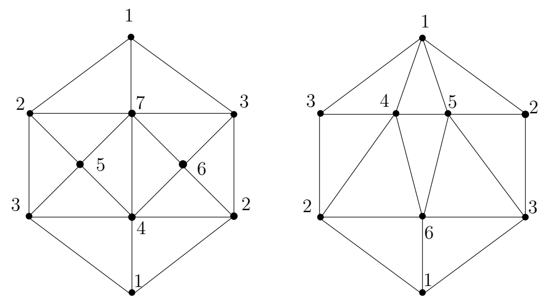

Figure 1: The two irreducible triangulations of $\mathbb{P}^2$.

A *diagonal flip* is an operation which replaces an edge e in the quadrilateral D formed by two faces sharing e with another diagonal of D (see Figure 2). If the resulting graph is not simple, then we do not apply it. Wagner [18] proved that any two triangulations on the plane with the same number of vertices can be transformed into each other by a sequence of diagonal flips, up to isotopy. This result has been extended to the torus [5], the real projective plane and the Klein bottle [16]. Moreover, Negami has proved that for any closed surface F^2, there exists a positive integer $N(F^2)$ such that any two triangulations G and G' on F^2 with $|V(G)| = |V(G')| \geq N(F^2)$ can be transformed into each other by a sequence of diagonal flips, up to homeomorphism [14]. Mori and Nakamoto [11] gave a linear upper bound of $(8n - 26)$ on the number of diagonal flips needed to transform one triangulation of $\mathbb{P}^2$ into another, up to isotopy. There are many papers concerning with diagonal flips in triangulations, see [15, 7] for more references.

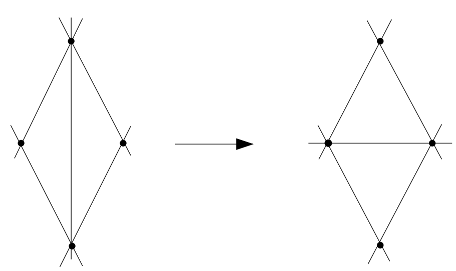

Figure 2: A diagonal flip.

In this paper, we address a different problem, which consists in computing a triangulation of the real projective plane, given a finite point set $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$ as input.

Definition 1.1 *Let us recall background definitions here. More extensive definitions are given for instance in [19, 9].*

- *An (abstract) simplicial complex is a set K together with a collection S of subsets of K called (abstract) simplices such that:*
  1. *For all $v \in K$, $\{v\} \in S$. The sets $\{v\}$ are called the vertices of K.*
  2. *If $\tau \subseteq \sigma \in S$, then $\tau \in S$.*

  Note that the property that $\sigma, \sigma' \in K \Rightarrow \sigma \cap \sigma' \leq \sigma, \sigma'$ can be deduced from this.
- *σ is a k-simplex if the number of its vertices is $k+1$. If $\tau \subset \sigma$, τ is called a face of σ.*
- *A triangulation of a topological space $\mathbb{X}$ is a simplicial complex K such that the union of its simplices is homeomorphic to $\mathbb{X}$.*

All algorithms known to compute a triangulation of a set of points in the Euclidean plane use the orientation of the space as a fundamental prerequisite. The projective plane is not orientable, thus none of these known algorithm can extend to $\mathbb{P}^2$.

We will always represent $\mathbb{P}^2$ by the *sphere model* where a point p is same as its diametrically opposite "copy" (as shown in Figure 3(a)). We will refer to this sphere as the *projective sphere*. A *triangulation* of the real projective plane $\mathbb{P}^2$ is a simplicial complex such that each face is bounded by a 3-cycle, and each edge can be seen as a greater arc on the projective sphere. We will also sometimes refer to a triangulation of the projective plane as a *projective triangulation*.

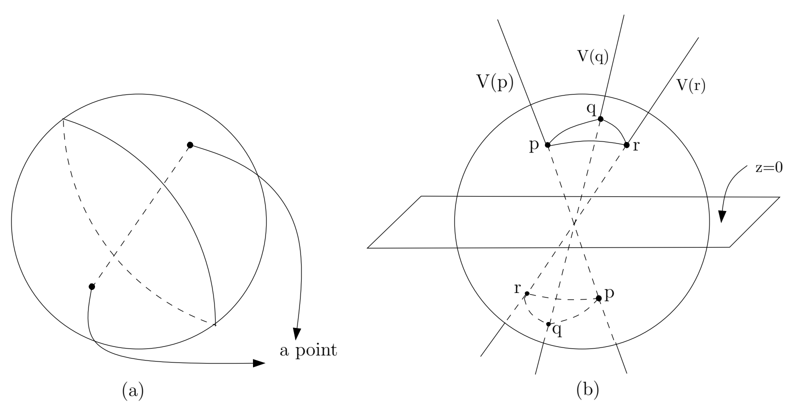

Figure 3: (a) The sphere model of $\mathbb{P}^2$. (b) $\triangle pqr$ separated from its "copy" by a *distinguishing plane* in $\mathbb{R}^3$.

Stolfi [17] had described a computational model for geometric computations: the *oriented projective plane*, where a point p and its diametrically opposite "copy" on the projective sphere are treated as two different points. In this model, two diametrically opposite triangles are considered as different, so, the computed triangulations of the oriented projective plane are actually not triangulations of $\mathbb{P}^2$. Identifying in practice a triangle and its opposite in some data-structure is not straightforward. Let us also mention that the oriented projective model can be pretty costly because it involves the duplication of every point, which can be a serious bottleneck on available memory in practice.

The reader should also note that obvious approaches like triangulating the convex hull of the points in $\mathcal{P}$ and their diametrically opposite "copies" (on the projective sphere) separately will not work: it may happen that the resulting structure is not a simplicial complex (see Figure 4 for the most obvious example), so, it is just not a triangulation (see definition 1.1).

1.1 Terminology and Notation

We assume that the positions of points and lines are stored as homogeneous coordinates in the real projective plane. Positions of points will be represented by triples (x, y, z) (with $z \neq 0$) and

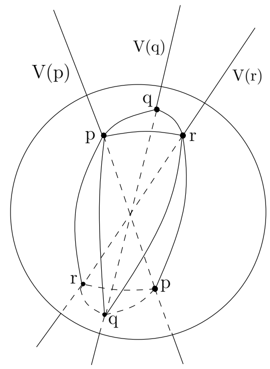

Figure 4: The convex hull of duplicated points is not a triangulation.

their coordinate vectors will be denoted by small letters like $p, q, r, \ldots$. Positions of lines will also be represented by triples $[x, y, z]$ but their coordinate vectors will be represented by capital letters like $L, M, N, \ldots$. We shall also state beforehand whether a given coordinate vector is that of a line or a point to avoid ambiguity. Point p and line M are incident if and only if the dot product of their coordinate vectors $p \cdot M = 0$. If p and q are two points then the line $L = pq$ can be computed as the cross product $p \times q$ of their coordinate vectors. Similarly the intersections of two lines M and N can be computed as the cross product $M \times N$ of their coordinate vectors.

We denote the line in $\mathbb{R}^3$ corresponding to a point p in $\mathbb{P}^2$ by $V(p)$. A plane in $\mathbb{R}^3$ which separates $\triangle abc$ from its diametrically opposite "duplicate copy" on the projective sphere will be referred to as a *distinguishing plane* for the given triangle (see Figure 3(b)). Note that a distinguishing plane is not unique for a given triangle. Also note that such a plane is defined only for non-degenerate triangles on the real projective plane.

1.2 Contents of the Paper

We first prove a necessary condition for the existence of a triangulation of the set $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$ of $\mathbb{P}^2$. More precisely, we show that such a triangulation always exists if at least six points in $\mathcal{P}$ are in *general position*, i.e., no three of them are collinear. So if the number of points n in $\mathcal{P}$ is very large, the probability of such a set of six points to exist is high, implying that it is almost always possible to triangulate $\mathbb{P}^2$ from a point set.

We design an algorithm for computing a projective triangulation of $\mathcal{P}$ if the above condition holds. The efficiency of the algorithm is not our main concern in this paper. The existence of an algorithm for computing a triangulation directly in $\mathbb{P}^2$ is our main goal. As far as we know, this is the first computational result on the real projective plane.

The paper is organized as follows. In section 2, we devise an "in-triangle" test for checking whether a point p lies inside a given $\triangle abc$. In section 3, we first prove that a triangulation of $\mathbb{P}^2$ always exists if at least six points in the given point set $\mathcal{P}$ are in general position. We then describe our algorithm for triangulating $\mathbb{P}^2$ from points in $\mathcal{P}$. Finally, in section 4, we present some open problems and future directions of research in this area.

2 The Notion of "Interior" in the Real Projective Plane

It is well-known that the real projective plane is a non-orientable surface. However, the notion of "interior" of a closed curve exists because the projective plane with a *cell* (any figure topologically equivalent to a disk) cut out is topologically equivalent to a Möbius band [9]. For a given triangle on the projective plane, we observe that its interior can be defined unambiguously if we associate a *distinguishing plane* with it. The procedure for associating such a plane with any given triangle will be described in Section 3. For now we will assume that we have been given $\triangle pqr$ along with

its distinguishing plane in $\mathbb{R}^3$. We further assume for simplicity that this plane is $z = 0$ for the given $\triangle pqr$ (as shown in Figure 3(b)). Consider the three lines $V(p), V(q)$ and $V(r)$ in $\mathbb{R}^3$. These lines give rise to four *double cones*, three of which are cut by the distinguishing plane. We define the *interior* of $\triangle pqr$ as the double cone in $\mathbb{R}^3$ which is not cut by its distinguishing plane. Based on the above definition, we define a many-one mapping $s : \mathbb{P}^2 \to \mathbb{R}^3$ from points in $\mathbb{P}^2$ to points in $\mathbb{R}^3$ as follows:

$$s(p) = s(x, y, z) = \begin{cases} (1, \frac{x}{z}, \frac{y}{z}) & z \neq 0; \\ (0, 1, \frac{y}{x}) & z = 0, x \neq 0; \\ (0, 0, 1) & z = 0, x = 0. \end{cases}$$

Given three points $a = (x_0, y_0, z_0), b = (x_1, y_1, z_1), c = (x_2, y_2, z_2)$ and a point $p = (x, y, z)$, p lies inside $\triangle abc$ if

$$sign \begin{vmatrix} s_0 \\ s_1 \\ s \end{vmatrix} + sign \begin{vmatrix} s_1 \\ s_2 \\ s \end{vmatrix} + sign \begin{vmatrix} s_2 \\ s_0 \\ s \end{vmatrix} = \pm 3 \tag{1}$$

and it lies on the perimeter of $\triangle abc$ if

$$sign \begin{vmatrix} s_0 \\ s_1 \\ s \end{vmatrix} + sign \begin{vmatrix} s_1 \\ s_2 \\ s \end{vmatrix} + sign \begin{vmatrix} s_2 \\ s_0 \\ s \end{vmatrix} = \pm 2 \tag{2}$$

Here $s_i = s(x_i, y_i, z_i)$ for $i = 0, 1, 2$, and $s = s(x, y, z)$. The function $sign(m)$ returns 1 if m is positive, 0 if m is zero, and -1 if m is negative. The reader should note that similar to the oriented projective model [17], there is no notion of interior when all a, b, c and p are at infinity. We now consider the case when the distinguishing plane in $\mathbb{R}^3$ is $\alpha x + \beta y + \gamma z = 0$, where α, β, γ are arbitrary constants. We use a linear transformation matrix $\mathcal{M}$ for transforming the given plane into the plane $z = 0$ according to the equation $\mathcal{M} \cdot p' = p$, where orientation is preserved. This transformation takes the coordinate vector p of a point to the vector p'. Now the s-mapping of equation (1) can be used for the "in-triangle" test with the new coordinate vectors, as described above. For the case when $\gamma \neq 0$, we have

$$\mathcal{M} = \begin{bmatrix} 0 & \frac{-(\beta^2+\gamma^2)}{\sqrt{(\beta^2+\gamma^2)(\alpha^2+\beta^2+\gamma^2)}} & \frac{\alpha}{\sqrt{(\alpha^2+\beta^2+\gamma^2)}} \\ \frac{\gamma}{\sqrt{\beta^2+\gamma^2}} & \frac{\alpha\beta}{\sqrt{(\beta^2+\gamma^2)(\alpha^2+\beta^2+\gamma^2)}} & \frac{\beta}{\sqrt{(\alpha^2+\beta^2+\gamma^2)}} \\ \frac{-\beta}{\sqrt{\beta^2+\gamma^2}} & \frac{\alpha\gamma}{\sqrt{(\beta^2+\gamma^2)(\alpha^2+\beta^2+\gamma^2)}} & \frac{\gamma}{\sqrt{(\alpha^2+\beta^2+\gamma^2)}} \end{bmatrix} \tag{3}$$

For the case when $\gamma = 0, \beta \neq 0$, we have

$$\mathcal{M} = \begin{bmatrix} \frac{\beta}{\sqrt{\alpha^2+\beta^2}} & 0 & \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} \\ \frac{-\alpha}{\sqrt{\alpha^2+\beta^2}} & 0 & \frac{\beta}{\sqrt{\alpha^2+\beta^2}} \\ 0 & -1 & 0 \end{bmatrix} \tag{4}$$

Finally, we have the case when $\gamma = \beta = 0$. In this case, we simply make the X-axis the new Y'-axis, the Y-axis the new Z'-axis, and the Z axis the new X'-axis. So our tranformation matrix $\mathcal{M}$ is as follows:

$$\mathcal{M} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{5}$$

Note that all the transformation matrices given by equations (2), (3) and (4) are *orthogonal* matrices, i.e., $\mathcal{M}^{-1} = \mathcal{M}^T$.

3 Computing the Projective Triangulation of a Point Set

We now proceed to discuss our algorithm for triangulating the real projective plane given a point set $\mathcal{P} = \{p_1, p_2, ..., p_n\}$ as input. We number the points in this section for diagramatic clarity. We first prove the following simple result for point sets in $\mathbb{P}^2$:

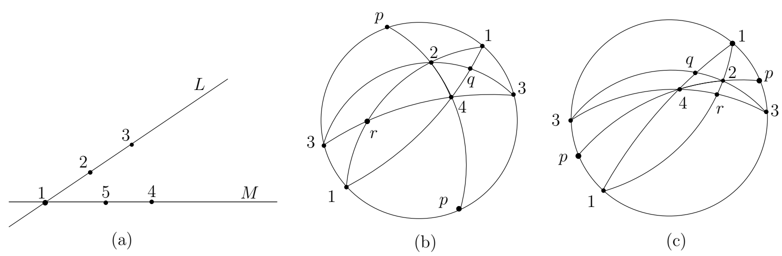

Figure 5: (a) Every point set $\mathcal{P}$ has a K_4-*quadrangulation*, unless $(n-1)$ points are collinear. (b) and (c) A K_4-quadrangulation is sufficient for triangulating the real projective plane.

Lemma 3.1 *If among every set of four points in the point set $\mathcal{P}$ at least three points are collinear, then at least $(n-1)$ points in $\mathcal{P}$ are collinear.*

Proof: It is easy to see that the lemma holds if all points in $\mathcal{P}$ are collinear. So we may safely assume that this is not the case. We prove the above lemma by the method of contradiction. Assuming that no set of $(n-1)$ points are collinear. Consider a set of four points $\{1, 2, 3, 4\}$ as shown in Figure 5(a). Since among every set of four points, at least three are collinear, so we assume that $1, 2$ and 3 are collinear. Now consider a fifth point 5 instead of 3, and assume that it does not lie on the line $L = 12$. From the given condition, it must lie on the line $M = 14$. But now no three points among the set $\{2, 3, 4, 5\}$ are collinear, a contradiction!

Corollary 3.1 *If $(n-1)$ points are not collinear in the given point set $\mathcal{P}$, then there exists a set of four points in $\mathcal{P}$, no three of which are collinear.*

We call such a set of four points a K_4-*quadrangulation*. Corollary 3.1 states that every point set $\mathcal{P}$ in which no $(n-1)$ points are collinear contains a K_4-quadrangulation. We now make the following important observation that such a point set can be used to construct a triangulation of the projective plane (see Figure 5(b,c)). We have the following lemma:

Lemma 3.2 *A K_4-quadrangulation can be used to construct a projective triangulation.*

Proof: Consider the points of the K_4-quadrangulation on the projective sphere and construct the lines (great circles) $\{12, 13, 14, 24, 23, 34\}$ (see Figure 5(b,c)). The intersection of these six lines define three more points $\{p, q, r\}$. We call these points *pseudo-points* because these may or may not be points in $\mathcal{P}$. It is now easy to see that the resulting triangulation is a simplicial complex and is isomorphic to the projective triangulation shown in Figure 1(a).

The reader should observe that every triangle in the above triangulation has precisely two copies on the projective sphere which are diametrically opposite (see Figure 5(b,c)). So it now becomes possible to associate a distinguishing plane with each triangle in the above triangulation unambiguously. For every $\triangle abc$ in the projective triangulation, we can take the plane through the center O of the projective sphere and parallel to the plane passing through the end-points a, b, c of one copy of $\triangle abc$. Given a query point u, we can now determine the triangle inside which it lies. We will use this fact quite extensively in our algorithm. The procedure described above is incomplete in the sense that we triangulate the real projective plane with the help of some pseudo-points. We now give a necessary condition for computing a projective triangulation from a point set $\mathcal{P}$. The reader should observe that every triangle in Figure 5(b,c) is incident with exactly one pseudo-point. We will refer to the set of triangles incident to one pseudo-point as a

region. Note that any two regions have the same set of vertices. For constructing a projective triangulation from $\mathcal{P}$ we will initially take help of pseudo-points, but we will go on deleting them as their use is over. We now present the following lemma:

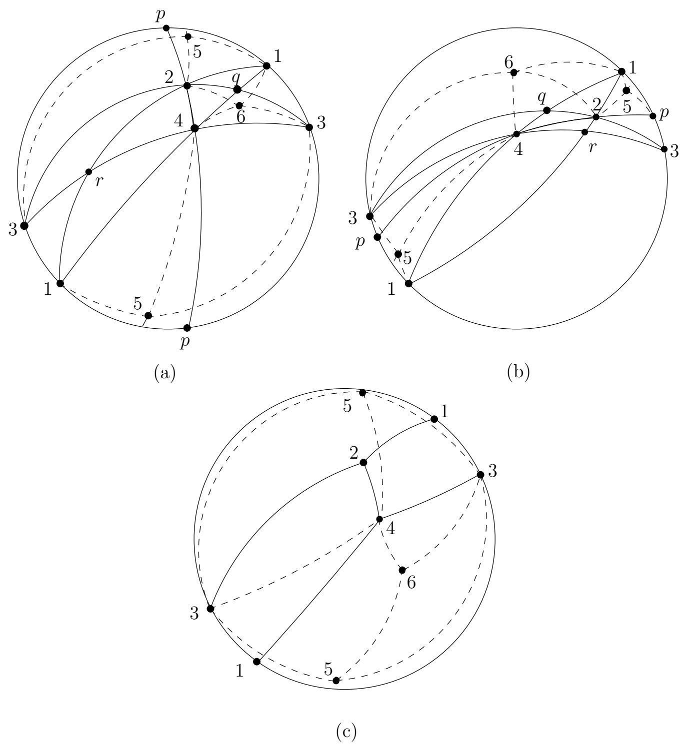

Figure 6: (a) and (b) Symmetric cases for constructing a projective triangulation. (c) A canonical set always exists when six points in $\mathcal{P}$ are in *general position*.

Lemma 3.3 *If there exists a set of six points (say,* $\{1,2,3,4,5,6\}$*) in a given point set* $\mathcal{P}$ *such that four of them (say,* $\mathcal{S} = \{1,2,3,4\}$*) form a* K_4*-quadrangulation and the other two (say,* $\{5,6\}$*) are in different regions of the projective triangulation formed by* $\mathcal{S}$*, then it is possible to triangulate the projective plane using these six points, unless* $(n-2)$ *points in* $\mathcal{P}$ *are collinear.*

Proof: We give a constructive proof of the above statement. We first construct a projective triangulation with the set $\{1,2,3,4\}$ (as described above). Suppose points 5 and 6 lie in the regions associated with the pseudo-points p and q respectively (see Figure 6(a,b)). We now add point 5 and make it adjacent to the vertices of its bounding region, deleting the pseudo-point p and the edges it was incident with. The newly added edges are shown by dashed lines. The pseudo-points have also been kept for better understanding. We now add point 6 and delete the corresponding pseudo-point q and the edges it was incident with. Now we intend to delete the pseudo-point r and construct a valid projective triangulation using only points in $\mathcal{P}$. Here we make the important observation that either the edge 12 or 34 can be flipped. To see this, note that if flipping of neither of these edges was possible, then 6 must lie to the "left" (as shown in Figure 6(a)) or "right" (as shown in Figure 6(b)) of both the lines 52 and 54, in which case flipping of edges would induce crossings. (Note that we refer to a point being on the "left" or "right" of a line only *locally* with respect to front half of the projective sphere.) However, the edge 24 lies in between these two lines and 6 cannot lie to its left (resp. right). Thus, our claim holds.

Suppose the edge 12 can be flipped. We then construct a valid projective triangulation by flipping 12, deleting the pseudo-point r and adding the edge 12 in that region. Observe that the projective triangulation constructed is isomorphic to that shown in Figure 1(b). In the event that

flipping of neither 12 nor 34 is possible, all four points $5, 2, 4, 6$ must be collinear. Since such a flip is also not possible with any other point in $\mathcal{P}$, they must all lie on the line 524, implying that $(n-2)$ of the points in $\mathcal{P}$ are collinear.

We will refer to such a K_4-quadrangulation which has two points of $\mathcal{P}$ in different regions as a *canonical set*. We now have almost all the basic tools required for triangulating the real projective plane from a point set $\mathcal{P}$. All that we need to characterize is the existence of a canonical set. So far we have not used anywhere the assumption that at least six points in $\mathcal{P}$ are in *general position*. It turns out that there always exists a canonical set in $\mathcal{P}$ in this case. We have the following lemma:

Lemma 3.4 *If at least six points in $\mathcal{P}$ are in general position, then there exists a canonical set.*

Proof: We prove this lemma by the method of contradiction. We assume that the lemma does not hold, so for every K_4-quadrangulation in $\mathcal{P}$, all other points of $\mathcal{P}$ are in the same region. Consider a K_4-quadrangulation $\{1, 2, 3, 4\}$ in $\mathcal{P}$. Suppose we add two more points 5 and 6, and they lie in the same region (as shown in Figure 6(c)). Now consider the K_4-quadrangulation formed by $\{4, 6, 3, 5\}$. If this is to satisfy the property that all points in $\mathcal{P}$ lie in exactly one of its regions, then it is easy to see that 2 must lie on or to the right of the line 54. But now 2 and 6 lie in different regions of the K_4-quadrangulation $\{4, 5, 1, 3\}$, a contradiction!

So we now have a procedure for triangulating the real projective plane given a point set $\mathcal{P}$ with at least six points in general position. We summarize our results in the following theorem:

Theorem 3.1 *Given a point set $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$ with at least six points in general position, it is always possible to construct a projective triangulation.*

We now present our algorithm which outputs a triangulation of the real projective plane given a point set $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$ with at least six points in general position.

1. Find a set $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ of six points such that no three points in $\mathcal{S}$ are collinear.
2. Construct a projective triangulation with the set $\mathcal{S}$. Associate distinguishing planes with every triangle of the triangulation.
3. **for** all points $p \in \mathcal{P} \backslash \mathcal{S}$ **do**
4. Identify the triangle $\triangle abc$ in which p lies.
5. Make p adjacent to the vertices a, b and c. Make the distinguishing plane of $\triangle apb, \triangle bpc$, and $\triangle cpa$ the same as that for $\triangle abc$.
6. **end for**
7. **return**(triangulation of $\mathbb{P}^2$).

There are two possible approaches for finding the set $\mathcal{S}$ in step 1. In the first approach, we arbitrarily choose a starting point q and initialize our set $\mathcal{S} = \{q\}$. For any point $p \in \mathcal{P} \backslash \mathcal{S}$, we add p in $\mathcal{S}$ if p is not collinear with any two points in $\mathcal{S}$. We stop when $\mathcal{S}$ contains six points. It may happen that we are not able to find such a set $\mathcal{S}$ of six points if we start with any random starting point q. So we iterate over all points in $\mathcal{P}$ for choosing the starting point. This approach has a worst-case time complexity of $\mathcal{O}(n^2)$. A slightly better approach can be adopted for performing step 1, which works in $\mathcal{O}(n)$ time if we assume that the minimum line cover of the point set $\mathcal{P}$ is greater than 4. In this approach, we first choose any two points 1 and 2. Let the line defined by them be L. We delete all other points in $\mathcal{P}$ on L. We now choose two more points 3 and 4. Let the line defined by them be M. We delete all other points in $\mathcal{P}$ on M. We also delete all other points on N (the line defined by 1 and 3) and T (the line defined by 2 and 4). Now choose two more points 5 and 6. We now have the required set $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$. It is easy to see that this approach takes $\mathcal{O}(n)$ time if the minimum line cover of $\mathcal{P}$ is greater than 4. The

above two approaches work reasonably well for most point sets. However, for certain point sets, it may happen that both these approaches fail to find such a set $\mathcal{S}$. We are currently unaware of an optimal method for finding such a set which works in all cases. We believe that some approach similar to that used for solving the "ordinary line" problem can be adopted for finding the same (see for instance [10, 12, 3]). After having found such a set $\mathcal{S}$, we find a canonical set within $\mathcal{S}$ by a procedure similar to that described in Lemma 3.4.

Once we have a canonical set, constructing the projective triangulation in step 2 takes $\mathcal{O}(1)$ time. We store the triangulation in a DCEL so that addition and deletion of edges and vertices takes $\mathcal{O}(1)$ time. The loop in steps 3-6 runs once for every point $p \in \mathcal{P} \backslash \mathcal{S}$. Inside the loop, we use our "in-triangle" test (as described in Section 2) for testing whether a point lies inside a given triangle. We use a procedure similar to that described by Devillers et al. in [4] for identifying $\triangle abc$ inside which p lies. We first choose any arbitrary vertex t of the current projective triangulation. We then identify the triangle whose interior is intersected by the line $L = tp$. This test is performed by checking for all edges E of all triangles sharing vertex t whether the intersection of L and the *line* described by E lies inside the given triangle (see Figure 7(a)). After having identified the starting triangle, we move to its neighbor sharing the edge E. In this way, we "walk" in the triangulation along the line L. We stop when p lies inside the current triangle. Although this method of "walking" in a triangulation has a worst-case time complexity of $\mathcal{O}(n)$, it is reasonably fast for most practical purposes. So the loop takes a total of $\mathcal{O}(n^2)$ steps. Thus, our algorithm computes a projective triangulation from a given point set $\mathcal{P}$ in $\mathcal{O}(n^2)$ steps.

As mentioned in the introduction, the complexity of the algorithm is not our main concern in the present paper. Still, note that our algorithm is incremental, which is an important property in practice. $\mathcal{O}(n^2)$ is a standard worst-case complexity for incremental algorithms computing triangulations in the Euclidean plane. After step 2, instead of inserting the points incrementally, we could do the following[1]: for each point, find the triangular face of the initial triangulation containing it. Then, in each of these faces, triangulate the set of points using the usual affine method. This can be done since the convex hulls of subsets of points in a triangular face of the initial triangulation can be defined with the help of distinguishing planes. This yields an optimal $O(n \log n)$ worst-case time (non-incremental) algorithm.

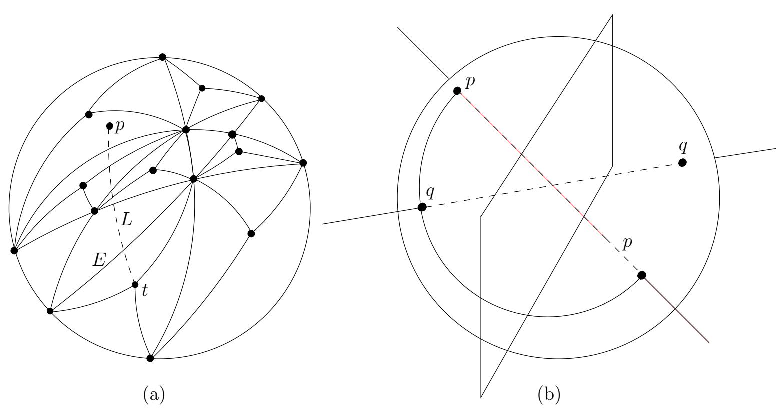

Figure 7: (a) "Walking" in a projective triangulation. (b) Two copies of the edge pq, only one is cut by the distinguishing plane.

4 Conclusion and Open Problems

It woud be interesting to check whether the metric on $\mathbb{P}^2$ allows to define a triangulation of the projective plane that would extend the notion of *Delaunay* triangulation, which is well-known in

[1] as suggested by an anonymous reviewer

the Euclidean setting. Then, extending the randomized incremental insertion with a hierarchical data-structure such as [2] to the projective case, if possible, would lead to an incremental algorithm with better theoretical (and practical, too) complexity.

Also, problems like the *Minimum Weight Triangulation* [13], *Minmax Length Triangulation* [6], etc., may have meaning even on the real projective plane. The *Minimum Weight Triangulation* problem was neither known to be NP-Hard nor solvable in polynomial time for a long time [8]. This open problem was recently solved and was shown to be NP-Hard by Mulzer and Rote [13]. The *Minmax Length Triangulation* problem asks about minimizing the maximum edge length in a triangulation of a point set $\mathcal{P}$. This problem was shown to be solvable in time $\mathcal{O}(n^2)$ by Edelsbrunner and Tan [6]. It would be interesting to analyze the complexity of these problems on the real projective plane $\mathbb{P}^2$.

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References

[1] D. W. Barnette. Generating triangulations of the projective plane. *J. Combin. Theory Ser. B*, 33:222–230, 1982.

[2] Olivier Devillers. The Delaunay hierarchy. *Internat. J. Found. Comput. Sci.*, 13:163–180, 2002.

[3] Olivier Devillers and Asish Mukhopadhyay. Finding an ordinary conic and an ordinary hyperplane. *Nordic J. Comput.*, 6:422–428, 1999.

[4] Olivier Devillers, Sylvain Pion, and Monique Teillaud. Walking in a triangulation. *Internat. J. Found. Comput. Sci.*, 13:181–199, 2002.

[5] A. K. Dewdney. Wagner's theorem for the torus graphs. *Discrete Math.*, 4:139–149, 1973.

[6] H. Edelsbrunner and T. S. Tan. A quadratic time algorithm for the minmax length triangulation. *SIAM J. Comput.*, 22:527–551, 1993.

[7] David Eppstein. Happy endings for flip graphs. In *Proc. 23rd Annual Symposium on Computational Geometry*, pages 92–101, 2007.

[8] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, New York, NY, 1979.

[9] M. Henle. *A Combinatorial Introduction to Topology*. W. H. Freeman, San Francisco, CA, 1979.

[10] L. Kelly and W. Moser. On the number of ordinary lines determined by n points. *Canad. J. Math.*, 10:210–219, 1958.

[11] R. Mori and A. Nakamoto. Diagonal flips in hamiltonian triangulations on the projective plane. *Discrete Math.*, 303:142–153, 2005.

[12] A. Mukhopadhyay, A. Agrawal, and R. M. Hosabettu. On the ordinary line problem in computational geometry. *Nordic J. Comput.*, 4:330–341, 1997.

[13] W. Mulzer and Günter Rote. Minimum weight triangulation is NP-hard. Technical Report B-05-23-revised, Freie Universität Berlin, july 2007. `http://arxiv.org/abs/cs/0601002`.

[14] S. Negami. Diagonal flips in triangulations on surfaces. *Discrete Math.*, 135:225–232, 1994.

[15] S. Negami. Diagonal flips in triangulations on closed surfaces, a survey. *Yokohama Math. J.*, 47:1–40, 1999.

[16] S. Negami and S. Watanabe. Diagonal transformations of triangulations on surfaces. *Tsubuka J. Math*, 14:155–166, 1990.

[17] J. Stolfi. *Oriented Projective Geometry: A Framework for Geometric Computations*. Academic Press, New York, NY, 1991.

[18] K. Wagner. Bemerkungen zum Vierfarbenproblem. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 46:26–32, 1936.

[19] Afra J. Zomorodian. *Topology for Computing*. Cambridge University Press, 2005.

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Geometric realization of a triangulation on the projective plane with one face removed

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Abstract

Let M be a map on a surface F^2. A *geometric realization* of M is an embedding of F^2 into a Euclidean 3-space $\mathbb{R}^3$ such that each face of M is a flat polygon. We shall prove that every triangulation G on the projective plane has a face f such that the triangulation of the Möbius band obtained from G by removing the interior of f has a geometric realization.

Keywords: triangulation, geometric realization, Möbius band, projective plane

1 Introduction

Let F^2 be a surface with at most one boundary component and let G be a map on F^2. If F^2 has a boundary, we suppose that some cycle of G coincides with the boundary of F^2. Such a cycle of G is called the *boundary* of G and denoted by ∂G. A *k-cycle* means a cycle of length k. A *triangulation* on F^2 is a map on F^2 such that each face is bounded by a 3-cycle. A vertex of G not on ∂G is called an *inner* vertex. (An *inner* vertex of a path means a vertex not on the ends.) Throughout this paper, we suppose that the graph of a map is *simple*, i.e., with no multiple edges and no loops. For a cycle or path C in G, a *chord* of C means an edge xy of G such that $x, y \in V(C)$ but $xy \notin E(C)$. A *k-wheel* is a triangulation on the disk such that its boundary is a k-cycle, called a *rim*, and there is only one inner vertex adjacent to the k vertices on the boundary. We often use symbols with subscripts to express vertices and so on, in which the subscripts are taken by suitable modulus.

A *geometric realization* of a map G on a surface F^2 is an embedding of F^2 into a Euclidean 3-space $\mathbb{R}^3$ such that each face of G is a flat polygon. We denote a geometric realization of G by $\hat{G}$. Steinitz's theorem states that a spherical map has a geometric realization if and only if its graph is 3-connected [10]. Moreover, Archdeacon et al. proved that every toroidal triangulation has a geometric realization [1]. In general, Grünbaum

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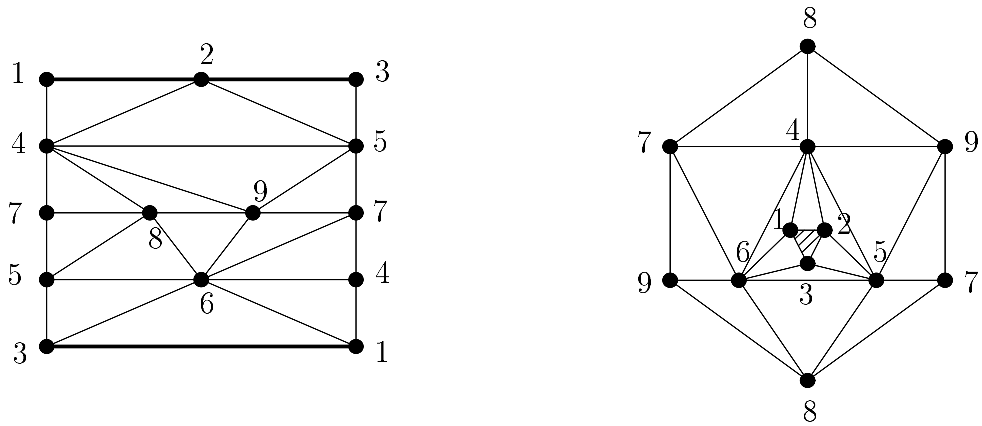


Figure 1: A Möbius triangulation with no geometric realization

conjectured that every triangulation on any orientable closed surface has a geometric realization [6], but Bokowski et al. recently showed that a triangulation by K_{12} on the orientable closed surface of genus 6 has no geometric realization [3]. (See also a good survey paper [5].)

Let us consider nonorientable surfaces, in particular, the projective plane. Since the projective plane itself is not embeddable in $\mathbb{R}^3$, no map on the projective plane has a geometric realization. However, the surface obtained from the projective plane by removing a disk, which is a Möbius band, is embeddable in $\mathbb{R}^3$, and hence we can expect that a triangulation on the Möbius band has a geometric realization. For simple notations, we call a triangulation on the projective plane and that of the Möbius band a *projective triangulation* and a *Möbius triangulation*, respectively.

In this paper, we consider geometric realizations of Möbius triangulations. However, Brehm [4] has already found a Möbius triangulation with no geometric realization shown in Figure 1, in which both express the same Möbius triangulation. (In Figure 1, identify the vertices with the same label. In the right side, the shaded part means the hole.)

Can we get an affirmative result for geometric realizations of Möbius triangulations? In this paper, answering such a question, we prove the following.

Let G be a projective triangulation and let f be a face of G. Let $G - f$ denote the Möbius triangulation obtained from G by removing the interior of f.

Theorem 1 *Every projective triangulation G has a face f such that the Möbius triangulation $G - f$ has a geometric realization.*

Why does Brehm's example have no geometric realization? We can prove that for each of its spatial embedding, the two disjoint 3-cycles 123 and 456 have a linking number at least 2. (See [9] for the definition of the linking number.) However, two 3-cycles with straight segments have linking number at most one, a contradiction. Hence we have the following fact.

Fact 2 *If a Möbius triangulation G has a boundary cycle C of length 3 and a 3-cycle C' disjoint from C which forms an annular region with C', then G never has a geometric realization.*

A graph G is said to be *cyclically k-connected* if G has no separating set $S \subset V(G)$ of G with $|S| \leq k-1$ such that each connected component of $G-S$ has a cycle.

Let G be a projective triangulation. By Fact 2, the cyclically 4-connectedness of G is necessary for a geometric realization of $G-f$ for any face f of G. We conjecture that the condition is also sufficient, as follows.

CONJECTURE 3 *Let G be a projective triangulation. Then G is cyclically 4-connected if and only if $G-f$ has a geometric realization for any face f of G.*

2 Irreducible Projective triangulations

Let G be a triangulation on a surface and let e be an edge of G. *Contraction* of e (or *contracting* e) is to remove e and identify the two endpoints of e. Furthermore, if this yields a digonal face bounded by a 2-cycle, then we replace the pair of parallel edges with a single edge. Let $[e]$ denote the vertex in the resulting graph obtained from e by its contraction. The inverse operation of the contraction of e is a *splitting* of $[e]$. We say that e is *contractible* if the map obtained from G by contracting e has no multiple edges and loops, and that G is *contractible* to T if G can be transformed into T by contractions of edges. An *irreducible* triangulation is one with no contractible edge.

See two projective triangulations $\mathcal{P}_1$ and $\mathcal{P}_2$ in Figure 2, in which we identify each pair of antipodal points of the hexagon. The graphs of $\mathcal{P}_1$ and $\mathcal{P}_2$ are isomorphic to K_6 and $K_4 + \bar{K}_3$, respectively, where $\bar{K}_n$ denotes a graph with n vertices and no edges. Let $\mathcal{M}_1$ be the Möbius triangulation obtained from $\mathcal{P}_1$ by removing a vertex and five faces incident to it, and let $\mathcal{M}_2$ be the one obtained from $\mathcal{P}_2$ by removing a vertex of degree 4 and four faces incident to it. The graph of $\mathcal{M}_1$ is isomorphic to K_5, and so $\mathcal{M}_1$ is called a *K_5-triangulation*. On the other hand, the graph of $\mathcal{M}_2$ includes a quadrangulation by K_4 with vertex set $\{1, 2, 3, 4\}$, which is called a *K_4-quadrangulation*.

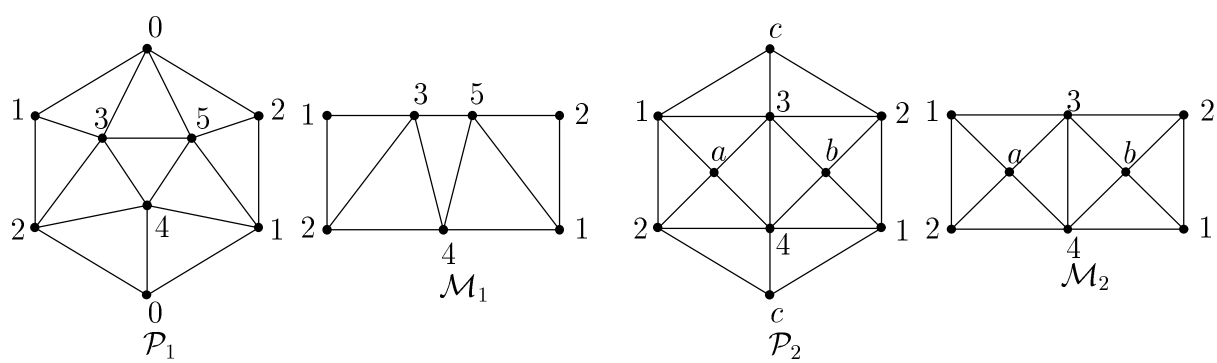


Figure 2: Irreducible projective triangulations $\mathcal{P}_1, \mathcal{P}_2$ and their submaps $\mathcal{M}_1, \mathcal{M}_2$

LEMMA 4 (Barnette [2]) *The projective plane admits exactly two irreducible triangulations, which are $\mathcal{P}_1$ and $\mathcal{P}_2$ shown in Figure 2.*

Let us construct a geometric realization of an irreducible projective triangulation with one face removed. For a point x and a straight segment e not intersecting x in $\mathbb{R}^3$, let $\Delta(x,e)$ denote the unique triangular disk in $\mathbb{R}^3$ containing x and the endpoints of e as its corners. Let M be a map and let $\hat{M}$ denote a geometric realization of M. Fix a point x in $\mathbb{R}^3$ not contained in $\hat{M}$. We say that an edge e of $\hat{M}$ is *(completely) seen* from x if $\Delta(x,e)$ intersects $\hat{M}$ only at e, and that an edge e of $\hat{M}$ is *half* seen from x if an interval e' of e containing one of the endpoints of e is completely seen from x.

PROPOSITION 5 *Let P be an irreducible projective triangulation. Then $P - f$ has a geometric realization for any face f of P.*

Proof. For each $\mathcal{P}_i$, we first construct a geometric realization of the Möbius triangulation $\mathcal{M}_i$ contained in $\mathcal{P}_i$. Figure 3 shows how to construct geometric realizations $\hat{\mathcal{M}}_1$ and $\hat{\mathcal{M}}_2$. In particular, we take the following 3-dimensional coordinates in $\mathbb{R}^3$:

- For $\hat{\mathcal{M}}_1$, $a = (0,0,8)$, $b = (-4,0,0)$, $c = (0,4,4)$, $d = (0,-4,4)$, $e = (4,0,0)$
- For $\hat{\mathcal{M}}_2$, $a = (-4,0,0)$, $b = (0,0,2)$, $c = (4,0,0)$, $d = (0,0,4)$, $e = (4,0,0)$ $p = (0,-4,1)$ and $q = (0,4,1)$.

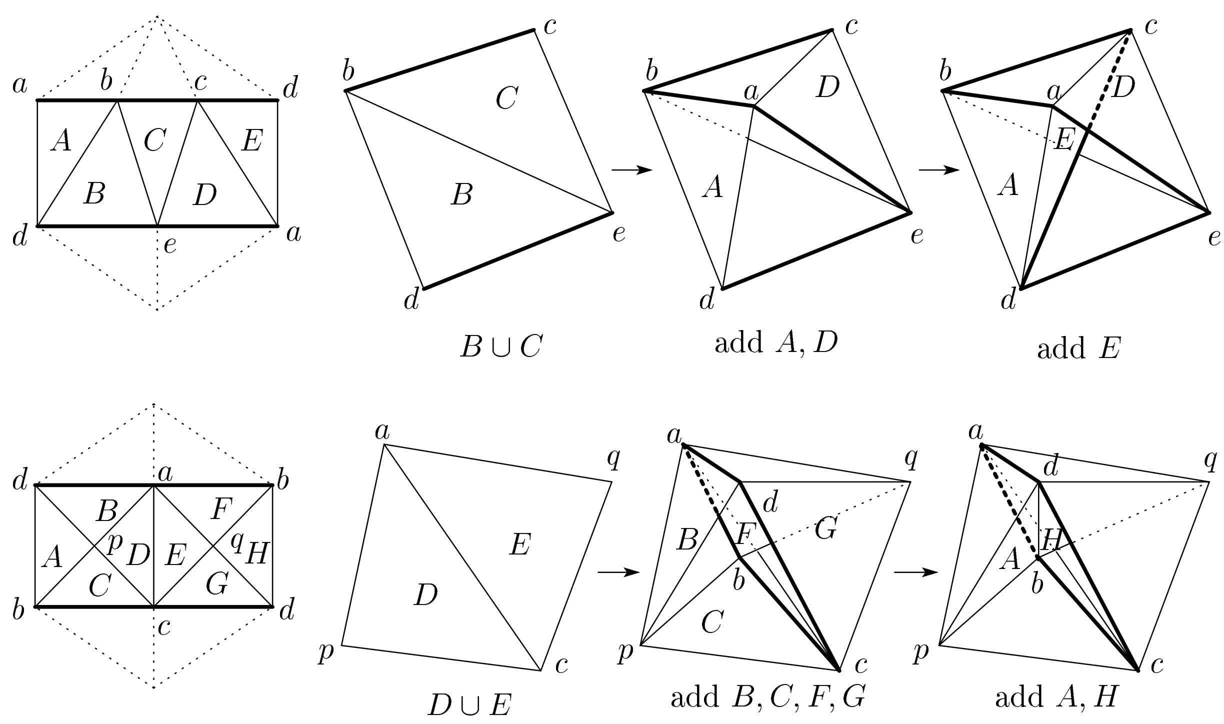


Figure 3: How to construct $\mathcal{M}_1$ and $\mathcal{M}_2$

In order to complete a geometric realization of $\mathcal{P}_i$ with one face removed, we have only to add four triangular disks incident to $\partial\hat{\mathcal{M}}_1$ and three triangular disks to $\partial\hat{\mathcal{M}}_2$. Observe that for $i = 1, 2$, there is only one edge on $\partial\hat{\mathcal{M}}_i$ which can not be completely seen from us in Figure 3. Such an edge in $\partial\mathcal{M}_1$ is cd, and the edge in $\partial\mathcal{M}_2$ is ab. Therefore, for $\hat{\mathcal{M}}_1$,

fixing a point x close to our eyes in $\mathbb{R}^3$ (for example, $x = (1, -1, 20)$), we can add four triangular disks $\Delta(x, ab)$, $\Delta(x, bc)$, $\Delta(x, de)$ and $\Delta(x, ea)$ with no internal collision of any two triangular disks. For $\hat{\mathcal{M}}_2$, fixing a fixed point $x = (1, -1, 20)$ in $\mathbb{R}^3$, for example, we can add three triangular disks $\Delta(x, bc)$, $\Delta(x, cd)$ and $\Delta(x, da)$. Thus, each $\mathcal{P}_i$ with some single face removed has a geometric realization.

It is easy to see that for any two faces f and f' of $\mathcal{P}_i$ $(i = 1, 2)$, there exists a homeomorphism over the projective plane carrying f into f', and hence each of $\mathcal{P}_1$ and $\mathcal{P}_2$ with any single face removed has a geometric realization. ■

REMARK 6 *In the geometric realization of $\mathcal{M}_1$ shown in Figure 3, b and c are contained in a half-space of $\mathbb{R}^3$ with respect to a plane containing a, d and e.* ■

3 Planar patches

A plane graph G with boundary walk C of length $m \geq 3$ is said to be a *near triangulation* and denoted by (G, C) if C is a cycle and if each inner face is a triangle. Let $v_0, \ldots, v_{k-1}$ be k distinct vertices, called the *nodes*, specified on ∂G in this cyclic order, where v_i and v_{i+1} are not necessarily adjacent in C, for each i. Let P_i be the path, called a *segment*, on C joining v_i and v_{i+1} but not containing v_{i+2}, for $i = 0, \ldots, k-1$. A near triangulation (G, C) is said to be *good* if no P_i has a chord.

LEMMA 7 *Let (G, C) be a near triangulation. If C has no chord, then $G - V(C)$ is connected or empty. Moreover, each vertex on C is incident to $G - V(C)$.*

Proof. We use induction on $|V(G)|$. If $|V(G)| = 3$, then the lemma clearly holds since $G - V(C)$ is empty. Hence we suppose that $|V(G)| \geq 4$. Let $v \in V(C)$ and let $P = v_1 \cdots v_k$ be the path of G consisting of the neighbors of v, where $v_1, v_k \in V(C)$. Since G has no chord, we have $k \geq 3$. Let $G' = G - v$, which is a near triangulation since G is chordless. Observe that G' might have chords incident to $v_2, \ldots, v_{k-1}$. Let $B_1, \ldots, B_m$ be the chordless near triangulations contained in G' sharing these chords one another, where $m \geq 1$. By induction hypothesis, the interior $\bar{B}_i$ of each B_i is connected or empty. Moreover, every B_i has some v_j on its boundary for $j \in \{2, \ldots, k-1\}$, and hence the vertex v_j is incident to $\bar{B}_i$. Therefore, $\bar{G} = G - V(C)$ is connected, since $V(\bar{G}) = \{v_2, \ldots, v_{k-1}\} \cup V(\bar{B}_1) \cup \cdots \cup V(\bar{B}_m)$. Clearly, every vertex on C is incident to $\bar{G}$, since C has no chord. ■

We often use the following lemma to find a suitable subgraph in a near triangulation or a Möbius triangulation. Let (G, C) be a near triangulation and let $x, y \in V(C)$. A path joining x and y and intersecting C only at x and y is called an *internal (x, y)-path*. The following immediately follows from Lemma 7.

LEMMA 8 *Let (G, C) be a near triangulation and let $x, y \in V(C)$ with $xy \notin E(C)$. There is an internal (x, y)-path in (G, C) if and only if there is no chord pq for any $p, q \in V(C) - \{x, y\}$ such that x, p, y, q appear on C in this cyclic order.* ■

An embedding $f : C \to \mathbb{R}^3$ is called an *exhibition* if $f(P_i)$ is a straight segment in $\mathbb{R}^3$ for $i = 0, \ldots, k-1$, and if $f(C)$ is projected to some hyperplane as a convex k-gon. Note that each vertex of C which is not a node is also fixed by f. This notion was first introduced in [1].

Lemma 9 *Let (G, C) be a good near triangulation with nodes $v_0, \ldots, v_{k-1}$ lying on C in this cyclic order, where $k \geq 3$, and let $f : C \to \mathbb{R}^3$ be an exhibition. Then $f(C)$ extends to a geometric realization of (G, C) contained in the interior of the convex hull of $f(C)$.*

Proof. We use induction on $|V(G)|$. If $|V(G)| = |V(C)| = 3$, then we have nothing to do. If $|V(C)| = 3$ and $|V(G)| \geq 4$, then (G, C) is a maximal plane graph, and $f(C)$ is contained in a plane in $\mathbb{R}^3$. It has already been proved in [11] that a maximal plane graph has a straight-line embedding on the plane with a given outer triangle.

So we suppose that $|V(C)| \geq 4$. Then we can take a chordless internal (x, y)-path P for some vertices $x, y \in V(C)$ belonging to distinct segments of C. (Such a P can be taken, by Lemma 8.) Consider two good near triangulations (G_1, C_1) and (G_2, C_2) such that $V(G_1) \cup V(G_2) = V(G)$ and $V(G_1) \cap V(G_2) = V(P)$. Now, regarding x and y as new nodes and joining x and y of $f(C)$ by a straight-line segment corresponding to P in $\mathbb{R}^3$, we can naturally obtain the exhibitions of C_1 and C_2, say $f(C_1)$ and $f(C_2)$. Then the convex hulls of $f(C_1)$ and $f(C_2)$ intersect only at P. By induction hypothesis, G_1 and G_2 have geometric realizations $\hat{G}_1$ and $\hat{G}_2$ extended from $f(C_1)$ and $f(C_2)$, which are contained in the convex hulls of $f(C_1)$ and $f(C_2)$, respectively. So, $\hat{G}_1 \cup \hat{G}_2$ is a geometric realization of G containing $f(C)$ and contained in the convex hull of $f(C)$. ■

Figure 4 shows an example of a near triangulation D and its geometric realization extended from an exhibition of ∂D.

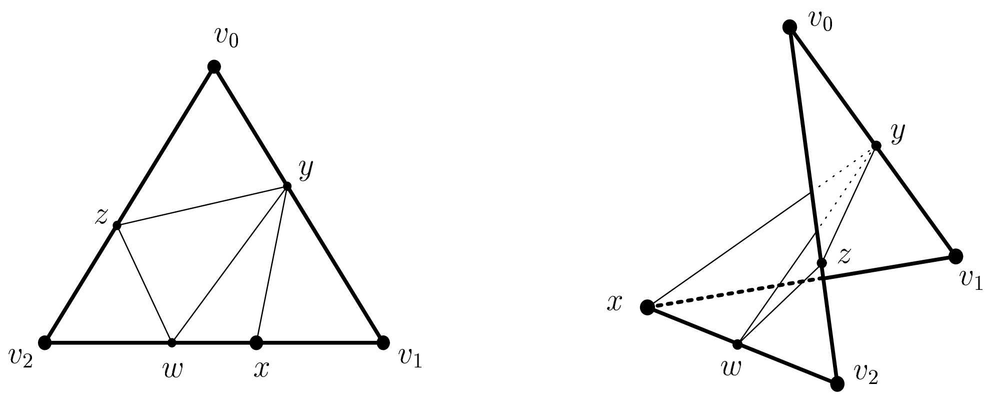


Figure 4: A geometric realization of D

Lemma 10 *Let (G, C) be a good near triangulation with nodes v_0, v_1, v_2, where $v_0v_1 \in E(C)$. Suppose that G has a face bounded by vv_0v_1, where $v \in V(G - C)$. Let D be any triangular disk with corners u_0, u_1, u_2. Then, for any inner point p of D, there is a straight-line embedding of G on D with each segment of C straight such that each v_i coincides with u_i, for $i = 0, 1, 2$, and that v coincides with p.*

Proof. Observe that G has a chordless path, say P, from v to some $u \in V(C) - V(P_0)$, since G has no chord v_0v_1. We may suppose that $u \in V(P_1) - \{v_1\}$. Then the three paths vv_0, vv_1, P decompose G into three good near triangulations. In D, even if we specify any inner point p as the position of v, we can choose a point q on the segment u_1u_2 of D as the position of u so that the three straight segments pu_0, pu_1, pq decompose D into three convex polygons. Since each boundary of the three polygons is an exhibition, we can apply Lemma 9 to them. ∎

Note that Lemma 10 does not necessarily hold if an embedding of C into ∂D is fixed. However, moving the positions of inner vertices of the segments v_1v_2 and v_0v_2 on C suitably, we can obtain a required embedding of G into D.

4 Projective triangulations with K_4-quadrangulations

We classify the projective triangulations into those with a K_4-quadrangulation as a subgraph and those not containing it. In this section, we deal with the former.

We first put an important lemma.

LEMMA 11 (Menger [7]) *Let G be a graph and let $v, v_1, \ldots, v_k$ be distinct vertices of G. Then G has k disjoint paths from v to v_i for $i = 1, \ldots, k$, if and only if there is no $S \subset V(G) - \{v, v_1, \ldots, v_k\}$ separating v and $\{v_1, \ldots, v_k\}$ in G such that $|S| < k$.*

LEMMA 12 *If a projective triangulation G has a K_4-quadrangulation K as a subgraph, then G has a subdivision of $\mathcal{P}_2$ containing K such that each path of G corresponding to an edge of $\mathcal{P}_2$ is chordless.*

Proof. Let K be a K_4-quadrangulation contained in G and let A_1, A_2 and A_3 be the three facial 4-cycles of K. Let $A_1 = abcd$ and let B_1 be the plane subgraph of G with boundary A_1.

We take an inner vertex v_1 of B_1 not contained in the interior of any separating 3-cycle of B_1, as follows. Since G is simple, we have $ac, bd \notin E(B_1)$, and hence B_1 has an inner vertex. If B_1 has no separating 3-cycle, then let v_1 be any inner vertex. Otherwise, let C be a separating 3-cycle of B_1 containing a maximal number of vertices in the interior. Since $ac, bd \notin E(B_1)$, at least one vertex of C, say v, is not contained in A_1. By the maximality of C, v is not surrounded by any separating 3-cycle. Therefore, let $v_1 = v$.

Then, by Lemma 11, there are four disjoint paths P_a, P_b, P_c, P_d from v_1 to a, b, c, d in B_1. Note that each of the four paths can be taken to be chordless if they are chosen to be shortest. Similarly, we can find a vertex v_i in the interior of A_i which has four disjoint paths to the four vertices of A_i, for $i = 2, 3$. Therefore, G contains a subdivision of $\mathcal{P}_2$ containing K. ∎

PROPOSITION 13 *If a projective triangulation G has a K_4-quadrangulation as a subgraph, then $G - f$ has a geometric realization for some face f of G.*

Proof. By Lemma 12, G has a subdivision of $\mathcal{P}_2$ containing a K_4-quadrangulation K. Let $V(K) = \{a, b, c, d\}$ and let $\{v_1, v_2, v_3\}$ be the vertices of G corresponding to $\bar{K}_3$, as in the above lemma. Suppose that v_1 is in the interior of the 4-cycle *abcd*. Let D denote the plane subgraph of G bounded by P_a, P_b, ab, where P_a and P_b are the chordless paths from v_1 to a and b, respectively. Let f be the face of D incident to ab. Let G' be the Möbius triangulation obtained from G by removing the interior of D. By Proposition 5 and Lemma 9, G' has a geometric realization $\hat{G}'$, since each disk corresponding to a face of $\mathcal{P}_2$ is a good near triangulation. Even if D consists of several faces of G, we can put $D - f$ in $\partial\hat{G}'$ avoiding a collision of faces, by Lemma 10, moving the positions of inner vertices of P_a and P_b suitably. ■

Lemma 14 (Mukae and Nakamoto [8]) *If a projective triangulation G has no K_4-quadrangulation as a subgraph, then G is contractible to $\mathcal{P}_1$.* ■

5 Split-K_5's in projective triangulations

Consider the irreducible projective triangulation $\mathcal{P}_1$ isomorphic to K_6. Let v be a vertex of $\mathcal{P}_1$ and let L_v be the link of v. Let G be a projective triangulation contractible to $\mathcal{P}_1$. Then G has a cycle, say C, which is contracted to L_v. We call C a *rim* in G. Clearly, cutting G along a rim C, we can obtain a Möbius triangulation G_M and a near triangulation G_D both of whose boundaries are C such that G_M is contractible to a K_5-triangulation $\mathcal{M}_1$, and G_D to a 5-wheel W_5.

Let $\mathcal{M}_1$ be a K_5-triangulation with vertices v_0, v_1, v_2, v_3, v_4, where $\partial\mathcal{M}_1 = v_0v_1v_2v_3v_4$ is called the boundary. Consider a splitting of a vertex v_i of $\mathcal{M}_1$. Then there are two ways of splitting v_i into two vertices v_i and v_i' of degree 3. The splitting of v_i is called a *boundary splitting* when both v_i and v_i' lie on the boundary, where we always suppose that v_i, v_i', v_{i+1} lie on $\partial\mathcal{M}_1$ in this order. Otherwise, it is called an *inner splitting*, where we always suppose that v_i lies on $\partial\mathcal{M}_1$. A *split-K_5* is a map H obtained from $\mathcal{M}_1$ by applying either a boundary splitting, an inner splitting, or no splitting to each vertex of $\mathcal{M}_1$, and subdividing edges. We call a vertex of H whose degree is greater than 2 is a *node* and a path in H containing only two nodes as its two endpoints a *segment*. Clearly, G_M contains a split-K_5 H as a subgraph so that $\partial H = \partial G_M$. We say that a vertex v of G_M is called a *boundary split node* (resp., an *inner split node*) if v is a node of H arisen by a boundary (resp., inner) splitting. Moreover, $\{v_i, v_i'\}$ is called a *boundary split pair* (resp., an *inner split node*) if they arose by a boundary (resp., inner) splitting. (See Figure 5.)

Throughout this paper, let $P(\alpha, \beta)$ denote the path of H contained in a segment joining two vertices α and β. The following is the main result in this section.

Lemma 15 *Let G be a projective triangulation contractible to $\mathcal{P}_1$. Then G contains a rim C separating G into a Möbius triangulation G_M and a near triangulation G_D both of whose boundaries are C such that*

(i) G_M has a split-K_5 H with either none or a single boundary split pair such that $\partial H = C$ (in the former case, the nodes of H lying on C are v_0, v_1, v_2, v_3, v_4, and in the latter case, those are $v_0, v_0', v_1, v_2, v_3, v_4$, where $\{v_0, v_0'\}$ is a boundary split pair), and

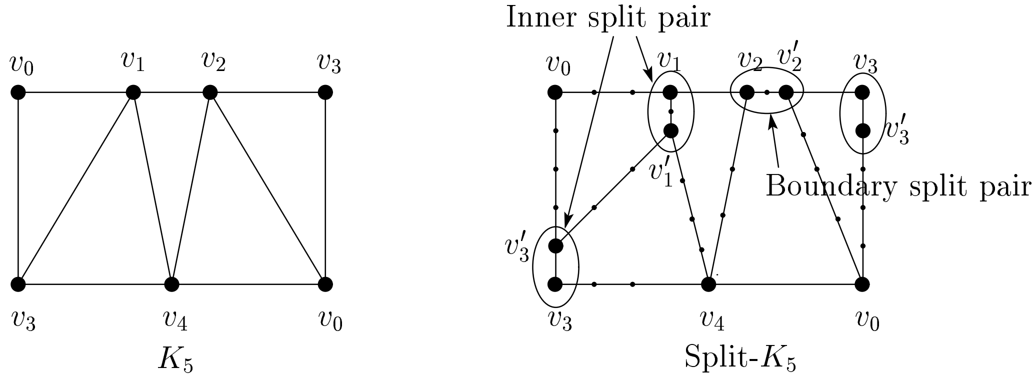


Figure 5: K_5 and split-K_5

(ii) in (G_D, C), for any fixed $k \in \{0,1,2,3,4\}$, there exists a vertex $v \in V(G_D - C)$ which has five chordless disjoint paths Q_i to v_i intersecting C only at v_i, for $i = 0,1,2,3,4$, such that Q_k has just an edge.

Proof. We first prove (i). Suppose that $\{a', a''\}$ is a boundary split pair arisen from a node a, as shown in Figure 6. Then a triangular region abc of $\mathcal{M}_1$ corresponds to a region Q with nodes a', a'', b, c. Note that if we take $P(a', a'')$ to be shortest in G, then Q has no edge yz with $y \in V(P(a', c)) - \{a'\}$ and $z \in V(P(a', a'')) - \{a'\}$. (For otherwise, taking z instead of a', we can take a smaller region corresponding to Q.) Therefore, by Lemma 8, we can take an internal (a', x)-path P for some x on either $P(a'', b) - \{a''\}$ or $P(b, c) - \{b, c\}$. In the former case, putting $H' = H - P(a'', x) \cup P$, we can decrease the number of boundary split pairs. In the latter case, consider $H' = H - P(a'', b) \cup P$, in which b can be regarded as a boundary split node but a is not. Hence we can gather several boundary split nodes into at most one node.

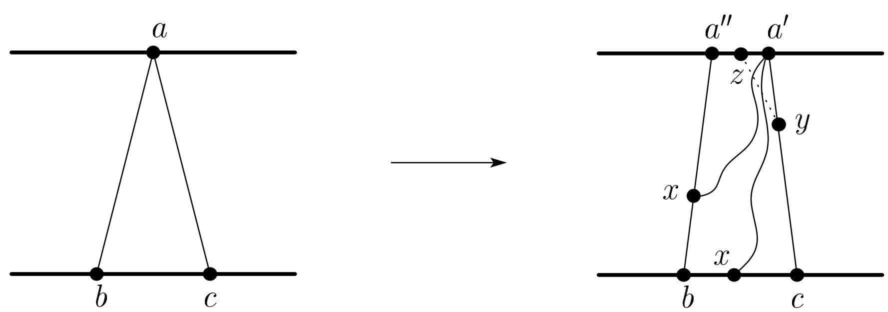


Figure 6: Move the position of a boundary split node

Next we consider (ii) in which we may suppose that $k = 0$, and that C is chosen in G so that G_D has a minimal number of vertices. Then, G_D has no chord. For a simple notation, we put $G_D = D$.

If (D, C) has only one inner vertex, then (D, C) is a wheel, by Lemma 7, and hence the lemma clearly holds. Therefore, (D, C) has at least two inner vertices. Pick up any neighbor v of v_0 with $v \notin V(C)$. This is possible since C has no chord in (D, C). Since v and v_0 are adjacent in D, we want to take the remaining four disjoint paths Q_i from v to

v_i such that Q_i intersects C only at v_i, for $i = 1, 2, 3, 4$. If this is impossible, one of the following holds for any v, by Lemma 11,

(1) There is $S \subset V(D-C) - \{v\}$ with $|S| \le 3$ separating $\{v\}$ from $\{v_1, v_2, v_3, v_4\}$ in D' for any v,

(2) there is $S \subset V(D-C) - \{v\}$ with $|S| \le 2$ separating $\{v, v_1\}$ from $\{v_2, v_3, v_4\}$ (or $\{v, v_4\}$ from $\{v_1, v_2, v_3\}$) in D' for any v, or

(3) there is $S \subset V(D-C) - \{v\}$ with $|S| \le 1$ separating $\{v, v_1, v_4\}$ from $\{v_2, v_3\}$, ($\{v, v_1, v_2\}$ from $\{v_3, v_4\}$, or $\{v, v_3, v_4\}$ from $\{v_1, v_2\}$) in D' for any v.

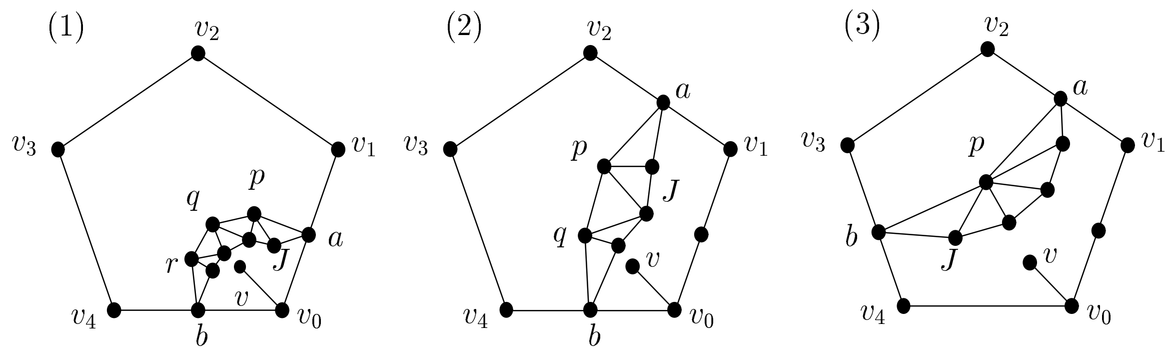


Figure 7: Structures of D

We shall prove that in each of the above cases, we can take another rim C' bonding a fewer number of vertices, and then getting a contradiction to the minimality of C.

(1) Let $S = \{p, q, r\}$. (The case when $|S| \le 2$ is similar.) Then there is a path $L = apqrb$ of length exactly four, where $a \in V(P_0) - \{v_0, v_1\}$ and $b \in V(P_4) - \{v_0, v_4\}$. Here we choose a and b to be nearest to v_0 on P_0 and P_4, respectively. See Figure 7(1). Take the unique path J joining a and b and consisting of neighbors of p, q, r lying on the side containing v_0 with respect to the cut L of (D, C), and let $C' = J \cup P(a, v_1) \cup P_1 \cup P_2 \cup P_3 \cup P(v_4, b)$. Then C' has no chord, and hence the near triangulation with boundary C' is a minor of a wheel with rim C', by Lemma 7. Moreover, specifying $b = v_0$ and adding $P(v_0, b)$ to H, we can take a split-K_5 with boundary C', contrary to the minimality of D. Now consider the case when there is v_0' on P_0. If v_0' is on $P(a, v_1)$, then specify v_0' as before, and otherwise, add the path $P(v_0', a)$ to H and specify $a = v_0'$.

(2) Let $S = \{p, q\}$. Similarly to the case (1), we can find a path $L = apqb$ of length exactly three, where $a \in V(P_1) - \{v_2\}$ and $b \in V(P_4) - \{v_0, v_4\}$. See Figure 7(2). Take the unique path J consisting of neighbors of p and q lying on the side containing v_0 with respect to L, and consider the smaller plane graph with no chord and including p and q. Then we can specify $a = v_1$ and $b = v_0$ and add $P(v_0, b)$ and $P(v_1, a)$ to H. If there is v_0' on P_0, then add $P(v_0', v_0)$ to H. Another case is similar.

(3) Take $S = \{p\}$ and consider the path $L = apb$, where $a \in V(P_1) - \{v_2\}$ and $b \in V(P_3) - \{v_3\}$. See Figure 7(3). Take the unique path J consisting of the neighbors of p lying on the side containing v_0 with respect to L, and consider the smaller plane graph with no chord and including p. Then we can specify $a = v_1$ and $b = v_4$. Moreover, since

D has no chord, we can find a path from v_0 to p intersecting C only at v_0, by Lemma 8, and hence we can specify one of inner vertices on $J - \{a, b\}$ as a new v_0. If there is v_0' on P_0, then add $P(v_0', v_0)$ to H. Other cases are similar. ■

6 Lemmas for geometric realization

Let H be a split-K_5 on the Möbius band and let v be a node of degree 4 on ∂H. Suppose that H has a geometric realization $\hat{H}$. Let A, B and C be three flat faces of $\hat{H}$ incident to v in this order. In $\hat{H}$, v is said to be of *type I* if C bends to the same side of A with respect to B, and of *type II* otherwise. (See Figure 8, where the left and the right show nodes v of type I and II, respectively.) In $\hat{\mathcal{M}}_1$ shown in Figure 5, only the vertex a is of type II and others are of type I.

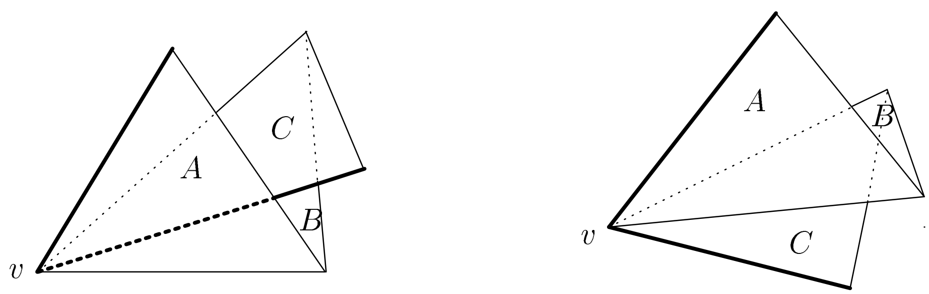


Figure 8: Vertices of degree 4 of type I and II

Suppose that a node v is of type I, where $A \cap B = e$ and $B \cap C = e'$, and $a = A \cap \partial H$ and $c = C \cap \partial H$. For $\hat{H}$, take a half-plane π_A parallel to A with boundary the straight line a but not containing e, and a half-plane π_C parallel to C with boundary the straight line c but not containing e'. It is easy to see that if π_A and π_C share a half-line, $\ell_{A,C}$, with endpoint v, then an inner splitting of v in $\hat{H}$ can be realized without changing the positions of all nodes of $\hat{H}$. Such a condition is called an *inner splitting condition* of v in $\hat{H}$. The following gives a criterion for a node of type I satisfying the inner splitting condition.

OBSERVATION 16 *A node v of type I satisfies an inner splitting condition if e and e' is contained in a half-space of $\mathbb{R}^3$ with respect to a plane containing a and c.* ■

See Figure 9, where (a) is an inner splitting and (b) is a boundary splitting of a vertex of type I. (On the other hand, the left of Figure 8 is a vertex of type I which does not satisfies an inner splitting condition, since π_C collides with A.)

Let f be a face of a geometric realization $\hat{H}$ such that the boundary of f contains exactly k nodes. We say that f is *convex* if f is a convex k-gon whose nodes are vertices of the k-gon f. We say that $\hat{H}$ is *convex* if each face of $\hat{H}$ is convex.

LEMMA 17 *Suppose that H' is obtained from H by a single inner splitting of a vertex v of degree 4, and H has a convex geometric realization $\hat{H}$. If v is of type I satisfying an inner splitting condition in $\hat{H}$, then H' has a convex geometric realization.*

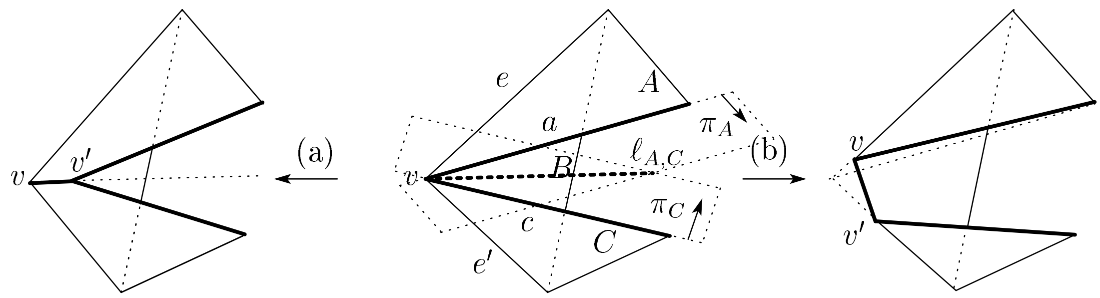


Figure 9: Inner and boundary splittings of a vertex of type I

Proof. Put v' on $\ell_{A,C}$ close to v, as shown in Figure 9. ■

LEMMA 18 *Suppose that H' is obtained from H by a single boundary splitting of a vertex v of degree 4, and H has a convex geometric realization $\hat{H}$. Then H' always has a convex geometric realization.*

Proof. Put v' on e', and move the position of v along e slightly, as shown in Figure 9. ■

7 Projective triangulations contractible to $\mathcal{P}_1$

In this section, we shall construct a geometric realization of a projective triangulation G contractible to $\mathcal{P}_1$ with one face removed. By Lemma 15, G has a split-K_5 H with boundary C, in which C has exactly five nodes v_0, v_1, v_2, v_3, v_4 or six nodes $v_0, v_0', v_1, v_2, v_3, v_4$. Let G_M and G_D be the Möbius triangulation and the good near triangulation obtained from G by cutting along C. Let P_i be the path on C joining v_i and v_{i+1}, not containing v_{i+2}, for $i = 0, 1, 2, 3, 4$.

LEMMA 19 *G_M has a geometric realization $\hat{G}_M$ such that each segment on C is a straight-line segment, and that from some fixed point x in $\mathbb{R}^3$, an edge, say e, on P_4 incident to v_0 is half seen, and all other edges on $\partial\hat{G}_M$ can be completely seen. In particular, if P_0 bends at v_0', then the convex hull of P_0 can be completely seen from x.*

Proof. For constructing a geometric realization of G_M, we use that of $\mathcal{M}_1$ shown in the top of Figure 3, where b and e are of type I satisfying the inner splitting condition, by Remark 6 and Observation 16. Note that a split-K_5 H in G_M has at most one boundary split pair on C, by Lemma 15.

Case 1. None of $v_0, \ldots, v_4$ are inner split nodes.

If C has no boundary split pair either, then H is homeomorphic to $\mathcal{M}_1$, and hence H has a geometric realization $\hat{H}$, as shown in the left of Figure 10. Note that $\hat{H}$ is convex since each face of H is triangular. In $\hat{H}$, fix the position of the vertices of C so that only one edge e of C can be half seen and all other edges of C can be completely seen from some point $x = (1, -1, 20)$ in $\mathbb{R}^3$. Since the boundary of each face of H gives an exhibition in $\mathbb{R}^3$, we can get a required geometric realization of G_M, by Lemma 9.

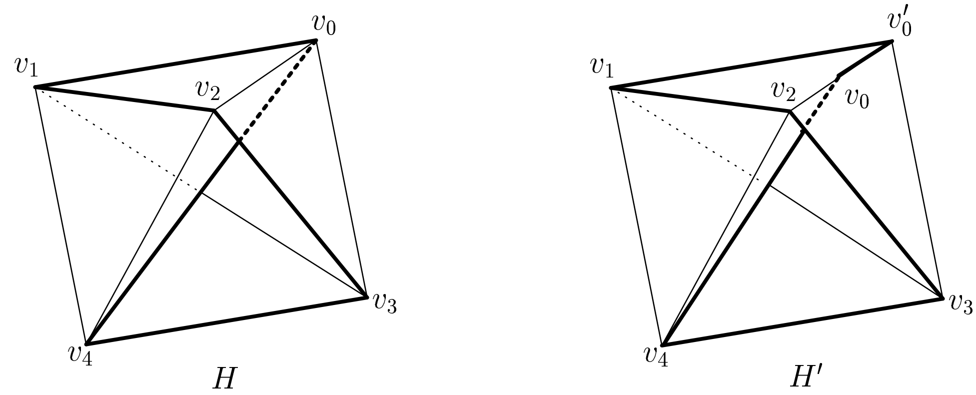


Figure 10: Geometric realizations of H and H'

Let H' be a split-K_5 obtained from the above H by a boundary splitting of v_0. By Lemma 18, a boundary splitting at v_0 can be applied freely so that the resulting geometric realization $\hat{H}'$ is convex. In particular, the convex hull of the bent P_0 in $\hat{H}'$ can be completely seen, since the segment v_0v_2 of $\hat{H}$ can be seen from $(1,-1,20)$. Therefore, by Lemma 9, G_M has a required geometric realization, as shown in the right of Figure 10.

Case 2. Both v_0 and v_4 are inner split nodes.

If each of v_1, v_2, v_3 is applied no splitting, then see Figure 11, where we give coordinates of the vertices, as follows:

$$\begin{array}{llll} v_0 = (-1,3,3), & v_0' = (1,4,4), & v_1 = (-4,0,0), & v_2 = (0,0,6), \\ v_3 = (4,0,0), & v_4 = (1,-3,3), & v_4' = (-1,-4,4) & \end{array}$$

Observe that two quadrilateral regions $v_1v_3v_4v_4'$ and $v_1v_3v_0'v_0$ of $\hat{H}$ are flat convex quadrilaterals in $\mathbb{R}^3$, since their xz-coordinates are $(-4,0)$, $(4,0)$, $(\pm 1,3)$, $(\mp 1,4)$. Let R be the pentagonal region $v_0v_0'v_2v_4'v_4$ of H. Then the embedding of ∂R in $\mathbb{R}^3$ is an exhibition, since the projection of R into the plane $x=0$ is a convex pentagon whose vertices have yz-coordinates: $(3,3)$, $(4,4)$, $(0,6)$, $(-4,4)$, $(-3,3)$. Hence, since other regions of H are triangular in $\mathbb{R}^3$, we get a geometric realization of G_M, by Lemma 9. However, seeing the body from $(1,-1,20)$, then the convex hull of ∂R hides P_4 completely, and hence the condition (ii) of the theorem might not be satisfied.

Hence we find an auxiliary path of G_M, as follows:

Claim. We may suppose that G_M has an internal (v_0,x)-path for R, denoted by P, for some vertex $x \in P(v_4,v_4') - \{v_4\}$. (See the left of Figure 11.)

Proof. We may suppose that H is chosen in G_M so that the number of inner vertices in R is minimal. By the symmetry, we shall prove that there is an internal (v_0,x)-path for some $x \in V(P(v_4,v_4')) - \{v_4\}$ or an internal (v_4,x')-path for some $x' \in V(P(v_0,v_0')) - \{v_0\}$. If neither of them exist, then we have an edge zw such that z is an inner vertex of P_4 and w is an inner vertex of $P(v_0',v_2) \cup P(v_2,v_4')$, by Lemma 8. By the symmetry, we may suppose that $w \in V(P(v_0',v_2)) - \{v_0'\}$. Let $H' = H - E(P(w,v_0')) \cup zw$, which is a split-$K_5$ with a boundary split node v_0. This is not the case. $\diamond$

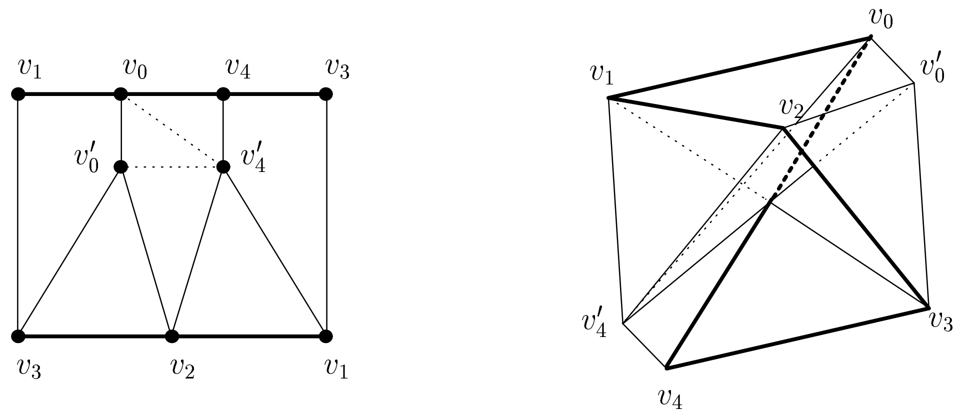


Figure 11: H with inner split nodes v_0 and v_4

Since the internal (v_0, x)-path divides R into a triangle xv_0v_4 and a region, say R', bounded by $v_0v_0'v_2v_4'x$, where we might have $x = v_4'$. Hence, take x as an internal point of the straight segment v_4v_4' in $\mathbb{R}^3$ and apply Lemma 9 to R'. Then the convex hull of $\partial R'$ no longer hides P_4 completely, if we see the body from $(1, -1, 20)$. (Figure 11 shows the case when $x = v_4'$ and P is an internal (v_0', v_4')-path.) Therefore, if we let H' be H with all edges of R' added, then H' has a required convex geometric realization.

Let's check that v_1 and v_3 satisfy the inner splitting condition in $\hat{H}'$, since they are nodes of type I. Since $\hat{H}'$ is symmetric, it suffices to deal with only v_3. For a plane π in $\mathbb{R}^3$ containing v_2, v_3 and v_4 (i.e., $\pi = \{(x, y, z) \in \mathbb{R}^3 : 3x - y + 2z = 12\}$), all other nodes lies in a half-space with respect to π. Therefore, by Observation 16, v_1 and v_3 must satisfy the inner splitting condition, and hence, even if v_1 or v_3 are inner split nodes in H', then the corresponding H' has a convex geometric realization, by Lemma 17.

Finally, we consider an inner splitting of v_2 in H'. We put $v_2' = (0, 0, 6 - \frac{1}{10})$, for example, and fix all other nodes as above. Then we can check that the regions bounded by $v_2v_2'v_4'v_1$ (or $v_2v_2'v_4'v_1'v_1$) and $v_2v_2'v_0'v_3$ (or $v_2v_2'v_0'v_3'v_3$) are exhibited, since they are projected to the plane $z = 0$ as a triangle or a convex quadrilateral. Therefore G_M has a required geometric realization, by Lemma 9.

Case 3. Other cases.

If exactly one of v_0 and v_4 is an inner split node, then we may suppose that v_4 is, by symmetry. Since the convex hull of $v_0v_2v_4'v_4$ does not hide P_4, we can deal with this case similarly to Case 2. (Figure 12 shows H with v_0 and v_4 applied a boundary splitting and an inner splitting respectively, and an example of its geometric realization.) The case when none of v_0 and v_4 are applied inner splittings can be dealt with similarly to Case 2, too. ■

The following is our main result in this section.

PROPOSITION 20 *If a projective triangulation G is contractible to a K_6-triangulation $\mathcal{P}_1$, then $G - f$ has a geometric realization for some face f of G.*

Proof. By Lemma 15, G contains a split-K_5 H with boundary C containing five nodes v_0, v_1, v_2, v_3, v_4 (or six nodes $v_0, v_0', v_1, v_2, v_3, v_4$), in which the near triangulation, denoted

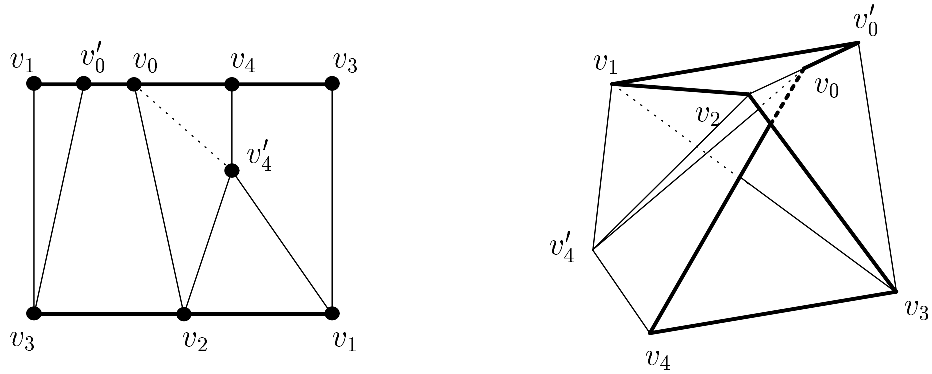


Figure 12: H with a boundary split node v_0 and an inner split node v_4

by (G_D, C), admits five chordless disjoint paths Q_i ($i = 0, 1, 2, 3, 4$) from some inner vertex v to v_i such that Q_0 is just an edge. These five paths divide G_D into five plane graphs, denoted by F_0, F_1, F_2, F_3, F_4, where each F_i is incident to Q_i and Q_{i+1}, for $i = 0, 1, 2, 3, 4$. Here we remove a face f bounded by vv_0p from F_4 incident to the unique edge of Q_0, and let $F_4' = F_4 - f$, where $p \notin \{v, v_0\}$ is a vertex of F_4. (See the right side of Figure 13.)

Let G_M denote the Möbius triangulation with boundary C. Then, by Lemma 19, G_M has a geometric realization $\hat{G}_M$ such that each P_i is a straight-line segment (only P_0 might bend), and that from some fixed point x in $\mathbb{R}^3$, an edge, say e, on P_4 incident to v_0 is half seen, and all other edges on $\partial \hat{G}_M$ can be completely seen. (See the left side of Figure 13.)

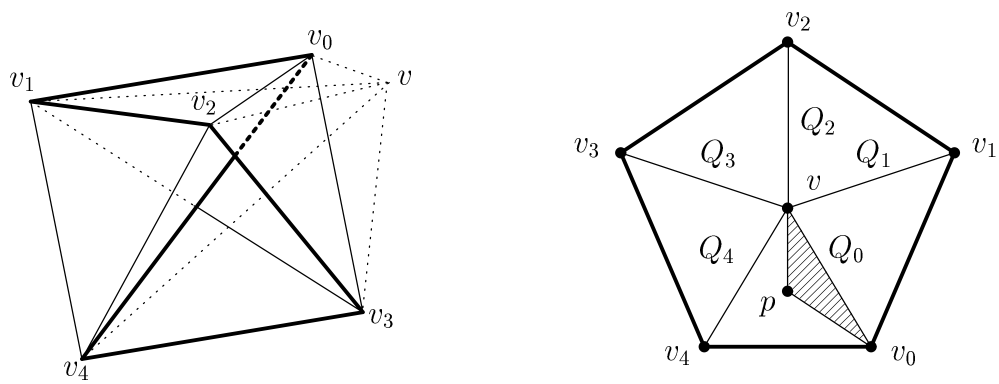


Figure 13: Pasting G_D with one face removed to $\partial \hat{G}_M$

Now, using the fixed point $x \in \mathbb{R}^3$, we first put F_4' in $\hat{G}_M$, as follows. If $p = v_4$, then we have $F_4 = f$, and hence we have nothing to do. If p is an inner vertex of F_4, then we put F_4' in $\hat{G}_M$ so that F_4' does not collide with the body, moving the position of inner vertices on P_4, by Lemma 10. If p is an inner point of P_4, then we apply Lemma 9. Since the edge e incident to f is half seen and all other edges on P_4 are completely seen from x, we can put F_4' in $\hat{G}_M$ naturally. If p is an inner point of Q_4, then fixing p on Q_4 in the position seeing v_0, we put F_4' in the body, by Lemma 9.

Finally, we shall put F_0, F_1, F_2, F_3 in the body of $G_M \cup F_4'$, making them be flat triangular disks by Lemma 9. (If P_0 bends at v_0', then the convex hull of v_0, v_0', v_1 can be

completely seen from v, by Lemma 19. Hence we can apply Lemma 9 to F_0, since the positions of x, v_0, v_0', v_1 of ∂F_0 gives an exhibition.) ■

8 Proof of the theorem

Proof of Theorem 1. Let G be a projective triangulation. By Lemma 14, either G contains a K_4-quadrangulation as a subgraph or it is contractible to K_6. In the former case, apply Proposition 13, and in the latter case, apply Proposition 20. ■

References

[1] D. Archdeacon, C.P. Bonnington and J.A. Ellis-Monanghan, How to exhibit toroidal maps in space, preprint.

[2] D.W. Barnette, Generating triangulations of the projective plane, *J. Combin. Theory Ser. B* **33** (1982), 222–230.

[3] J. Bokowski and A. Guedes de Oliveira, On the generation of oriented matroids, *Discrete Comput. Geom.* **24** (2004), 197–208.

[4] U. Brehm, A nonpolyhedral triangulated Möbius strip, *Proc. Amer. Math. Soc.* **89** (1983), 519–522.

[5] U. Brehm and J.M. Wills, Polyhedral manifolds, in "Handbook of Convex Geometry", pp. 535–554, (eds P.M. Gruber and J.M. Wills), North Holland, Amsterdam, 1993.

[6] B. Grünbaum, "Convex polytopes", *Pure and Applied Mathematics* Vol. 16, Interscience-Wiley, New York, 1967.

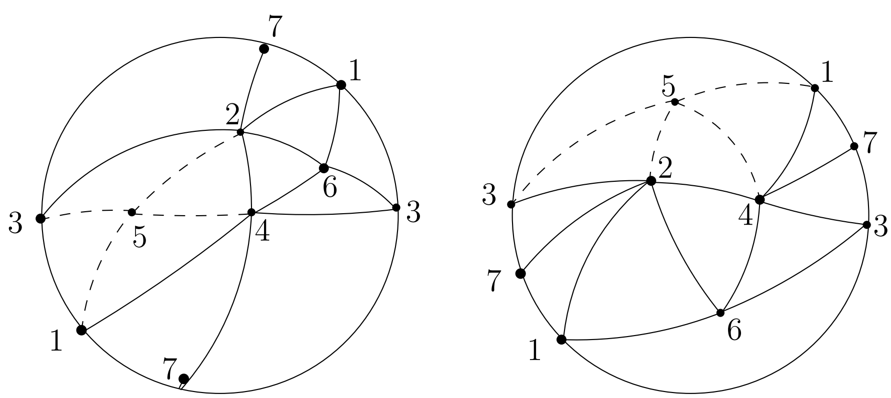
[7] K. Menger, Zur allgemeinen Kurventheorie, *Fund. Math.* **10** (1927), 95–115.

[8] R. Mukae and A. Nakamoto, K_6-Minors in triangulations and complete quadrangulations, preprint.

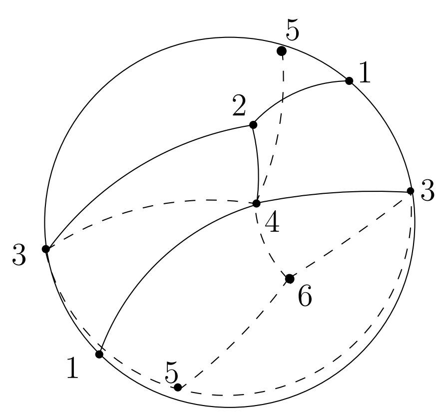
[9] D. Rolfsen, *Knot and links*, Math. Lecture Series 7, Publish or Perish, Berkeley, Calif., 1976.

[10] E. Steinitz, Polyeder un Raumeinteilunger, *Enzykl. Math. Wiss.* Vol. 3, Teil 3A612 (1922), 1–139.

[11] W.T. Tutte, How to draw a graph, *Proc. London Math. Soc.*, **13** (1963), 743–767.



(a) (b)



(c)